The Vasicek Interest Rate Process
Part IV - Interest Rate Curves

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In this white paper we will use the interest rate equations from **The Vasicek Interest Rate Process - Parts I, II and III** to build (1) the yield curve, (2) the forward curve and (3) the swap curve. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

The go-forward interest rate assumptions from Parts I and II were...

Description	Symbol	Value
Current short rate	r_s	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

We are tasked with answering the following questions...

Question 1: Graph the yield and forward rate curves over the time interval $[0, 10]$.

Question 2: Graph the swap rate curve over the time interval $[0, 10]$.

Bond Price

We will define the variable r_t to be the short rate at time t and the variable λ to be the rate of mean reversion. In Part I we determined that the equation for the mean of the random short rate at time t as a function of the known short rate at time s was... [1]

$$r_t \text{ mean} = r_\infty + (r_s - r_\infty) \operatorname{Exp}\left\{ - \lambda\,(t-s)\right\} \tag{1}$$

We will define the variable σ to be short rate volatility. In Part I we determined that the equation for the variance of the random short rate at time t as a function of the known short rate at time s was... [1]

$$r_t \text{ variance} = \frac{1}{2}\,\sigma^2 \left(1 - \operatorname{Exp}\left\{ - 2\,\lambda\,(t-s)\right)\lambda^{-1} \tag{2}$$

We will define the variable $R_{s,t}$ to be the cumultive short rate over the time interval $[s, t]$. The equation for the cumulative short rate is... [2]

$$R_{s,t} = \int\limits_s^t r_u\,\delta u \tag{3}$$

In Part II we determined that the equation for the mean of the random cumulative short rate was... [2]

$$R_{s,t} \text{ mean} = r_\infty\,t + (r_\infty - r_s)\left(\operatorname{Exp}\left\{ - \lambda\,(t-s)\right\} - 1\right)\lambda^{-1} \tag{4}$$

In Part II we determined that the equation for the variance of the random cumultive short rate over the time interval $[s,t]$... [2]

$$R_{s,t} \text{ variance} = \frac{1}{2}\,\sigma^2 \Bigg(2\,\lambda\,(t-s) - 3 + 4\,\mathrm{Exp}\Bigg\{ -\lambda\,(t-s) \Bigg\} - \mathrm{Exp}\Bigg\{ -2\,\lambda\,(t-s) \Bigg\} \Bigg) \lambda^{-3} \tag{5}$$

We will define the variable $B(s,t)$ to be price at time s of a default-free, zero-coupon bond that pays FV (face value) at some future time t. Using the equations above the equation for bond price at time s is... [3]

$$B(s,t) = FV\,\mathrm{Exp}\Bigg\{ -\int\limits_s^t r_x\,\delta x \Bigg\} = FV\,\mathrm{Exp}\Bigg\{ -R_{s,t} \text{ mean} + \frac{1}{2}\,R_{s,t} \text{ variance} \Bigg\} \tag{6}$$

The Yield Curve

We will define the variable $Y(s,t)$ to be annualized bond yield over the time interval $[s,t]$. Note that we can rewrite bond price Equation (6) above as...

$$B(s,t) = FV\,\mathrm{Exp}\Bigg\{ -Y(s,t)\,(t-s) \Bigg\} \tag{7}$$

Taking the log of both sides of Equation (7) above we get the following equation...

$$\ln(B(s,t)) + Y(s,t)\,(t-s) = \ln(FV) \tag{8}$$

Solving Equation (8) above for $Y(s,t)$ we get the following equation for annualized yield...

$$Y(s,t) = \frac{\ln(FV) - \ln(B(s,t))}{t-s} \quad \text{...when...} \quad t > s \tag{9}$$

Note that when $t=s$ in Equation (9) above the equation for bond yield is undefined (i.e. denominator is zero). In this case we will use l'Hopital's rule to calculate bond yield. We will define the functions $f(t)$ and $g(t)$ as follows...

$$f(t) = \ln(FV) - \ln(B(s,t) \quad \text{...and...} \quad g(t) = t - s \tag{10}$$

Using Appendix Equation (30) below the derivative of Equation (10) above with respect to the time variable t is...

$$\frac{\delta}{\delta t}\,f(t) = r_t \quad \text{...and...} \quad \frac{\delta}{\delta t}\,g(t) = 1 \tag{11}$$

Using Equation (11) above the equation for bond yield at time s is...

$$Y(s,s) = \lim_{t \to s} Y(s,t) = \frac{f'(t)}{g'(t)} = \frac{r_t}{1} = r_t \tag{12}$$

The Forward Rate Curve

Using bond price Equation (6) above we have the following two default-free, zero-coupon bonds purchased at time s that pay FV (face value) at future times t and u...

$$B(s,t) = FV\,\mathrm{Exp}\Bigg\{ -\int\limits_s^t r_x\,\delta x \Bigg\} \quad \text{...and...} \quad B(s,u) = FV\,\mathrm{Exp}\Bigg\{ -\int\limits_s^u r_x\,\delta x \Bigg\} \quad \text{...where...} \quad t < u \tag{13}$$

Note that we can rewrite the equation for the price of the longer-term bond ($B(s,u)$ in Equation (13) above) in terms of the shorter-term bond ($B(s,t)$) ...

$$\begin{aligned} B(s,u) &= FV\,\mathrm{Exp}\Bigg\{ -\int\limits_s^u r_x\,\delta x \Bigg\} \\ &= FV\,\mathrm{Exp}\Bigg\{ -\int\limits_s^t r_x\,\delta x \Bigg\}\,\mathrm{Exp}\Bigg\{ -\int\limits_t^u r_x\,\delta x \Bigg\} \\ &= B(s,t)\,\mathrm{Exp}\Bigg\{ -\int\limits_t^u r_x\,\delta x \Bigg\} \end{aligned} \tag{14}$$

We will define the variable $F(s,t,u)$ to be the annualized forward rate contracted at time s on a risk-free investment purchased at time t and matures at time u. Using Equation (14) above the equation for the forward rate is...

$$F(s,t,u) = \frac{1}{u-t} \int\limits_t^u r_x \, \delta x \text{ ...where... } s < t < u \tag{15}$$

Using Equation (15) above we can rewrite Equation (14) above as...

$$B(s,u) = B(s,t) \, \mathrm{Exp}\left\{ -F(s,t,u)\,(u-t) \right\} \tag{16}$$

Taking the log of both sides of Equation (16) above we get the following equation...

$$\ln(B(s,u)) = ln(B(s,t)) - F(s,t,u)\,(u-t) \tag{17}$$

Solving Equation (17) above for $F(s,t,u)$ we get the following equation for the forward rate of interest over the time interval $[t,u]$...

$$F(s,t,u) = \frac{\ln(B(s,t)) - \ln(B(s,u))}{u-t} \text{ ...where... } s < t < u \tag{18}$$

The Swap Curve

A fixed-for-floating swap is a contractual arrangement between two parties in which one party swaps the interest cash flows of fixed-rate loan(s) with those of floating-rate loan(s) held by another party. The swap rate denotes the fixed rate that a party to a swap contract receives in exchange for the obligation to pay a short-term rate, such as the Labor or Federal Funds rate. A fixed-for-floating swap allows one to better match assets and liabilities that are sensitive to interest rate movements.

We are currently standing at time s and want to invest capital over the time interval $[t,u]$ where $s < t < u$. To prevent arbitrage one should be able to invest one dollar at the variable rate (the forward rate) and one dollar at the fixed rate (the swap rate) at time t and end up with the same dollar amount at time u. If the dollar amounts are different at time u then there is an arbitrage opportunity.

To prevent arbitrage we want to solve the following equation for the fixed swap rate $S(s,t,u)$...

$$\mathrm{Exp}\left\{ S_{s,t,u}\,(u-t) \right\} = \prod_{n=1}^{(u-t)\Delta} \mathrm{Exp}\left\{ F_{s,\frac{n-1}{\Delta},\frac{n}{\Delta}} \right\} \text{ ...where... } \Delta = \text{number of annual compounding periods} \tag{19}$$

Equation (19) above says that if I invest \$1.00 at the fixed rate over the time interval $[t,u]$ and \$1.00 at the forward rate over the time interval $[t,u]$ then I should end up with the same amount of money at time u.

The Answers To Our Hypothetical Problem

Question 1: Graph the yield and forward rate curve over the time interval $[0,10]$.

Yield calculation example: Using Equations (4), (5) and (6) above and the model assumptions in the table above the price of a risk-free bond due at the end of month 36 is...

$$\begin{aligned}
\text{mean} &= 0.09 \times \frac{36}{12} + (0.09 - 0.04) \times \left(\mathrm{Exp}\left\{ -0.35 \times \frac{36}{12} \right\} - 1 \right) \times 0.35^{-1} \\
&= 0.17713 \\
\text{variance} &= \frac{1}{2} \times 0.03^2 \times \left(2 \times 0.35 \times \frac{36}{12} - 3 + 4 \times \mathrm{Exp}\left\{ -0.35 \times \frac{36}{12} \right\} - \mathrm{Exp}\left\{ -2 \times 0.35 \times \frac{36}{12} \right\} \right) \times 0.35^{-3} \\
&= 0.00396 \\
\text{price} &= 1{,}000 \times \mathrm{Exp}\left\{ -0.17713 + \frac{1}{2} \times 0.00396 \right\} \\
&= 839.33
\end{aligned} \tag{20}$$

Using Equation (9) above and the bond price in Equation (20) above the yield calculation for our example month 36 is...

$$Y(0, \tfrac{36}{12}) = \frac{\ln(1,000.00) - \ln(839.33)}{\frac{36}{12}} = 0.0584 \tag{21}$$

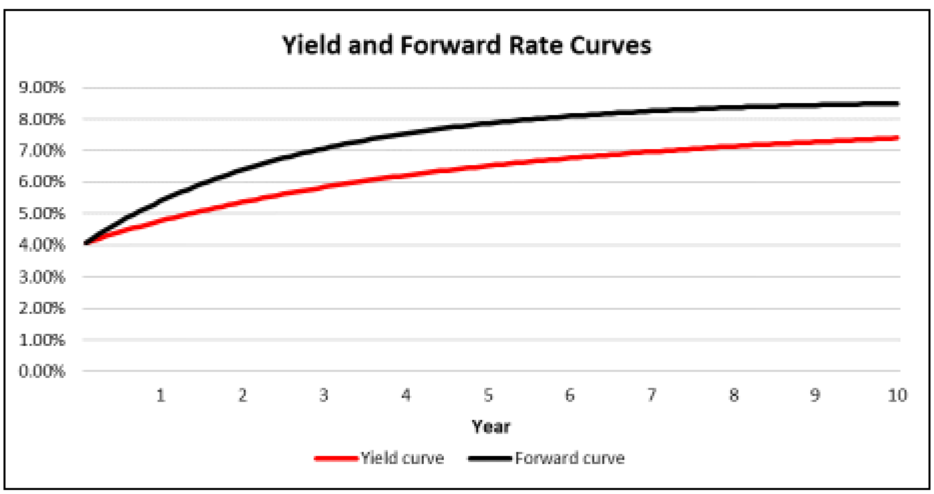
Forward rate example: Using Equations (4), (5) and (6) above and the model assumptions in the table above the price of a risk-free bond due at the end of month 35 is...

$$\begin{aligned}
\text{mean} &= 0.09 \times \frac{35}{12} + (0.09 - 0.04) \times \Bigg(\text{Exp}\Bigg\{ -0.35 \times \frac{35}{12} \Bigg\} - 1\Bigg) \times 0.35^{-1} \\
&= 0.17111 \\
\text{variance} &= \frac{1}{2} \times 0.03^2 \times \Bigg(2 \times 0.35 \times \frac{35}{12} - 3 + 4 \times \text{Exp}\Bigg\{ -0.35 \times \frac{35}{12}\Bigg\} - \text{Exp}\Bigg\{ -2 \times 0.35 \times \frac{35}{12}\Bigg\}\Bigg) \times 0.35^{-3} \\
&= 0.00371 \\
\text{price} &= 1,000 \times \text{Exp}\Bigg\{ -0.17111 + \frac{1}{2} \times 0.00371 \Bigg\} \\
&= 844.29
\end{aligned} \tag{22}$$

Using Equation (18) above and the bond price equations (20) and (25) above the forward rate calculation for our example month 36 is...

$$F(0, \tfrac{35}{12}, \tfrac{36}{12}) = \frac{\ln(844.29) - \ln(839.33)}{\frac{36}{12} - \frac{35}{12}} = 0.0707 \tag{23}$$

Using the example calculations above as our guide the graph of the yield and forward rate curve over the time interval $[0, 10]$ is...



Question 2: Graph the swap rate curve over the time interval $[0, 10]$.

Yield calculation example: Using Equations (4), (5) and (6) above and the model assumptions in the table above the price of a risk-free bond due at the end of month 60 is...

$$\begin{aligned}
\text{mean} &= 0.09 \times \frac{60}{12} + (0.09 - 0.04) \times \left(\text{Exp}\left\{-0.35 \times \frac{60}{12}\right\} - 1\right) \times 0.35^{-1} \\
&= 0.33197 \\
\text{variance} &= \frac{1}{2} \times 0.03^2 \times \left(2 \times 0.35 \times \frac{60}{12} - 3 + 4 \times \text{Exp}\left\{-0.35 \times \frac{60}{12}\right\} - \text{Exp}\left\{-2 \times 0.35 \times \frac{60}{12}\right\}\right) \times 0.35^{-3} \\
&= 0.01223 \\
\text{price} &= 1,000 \times \text{Exp}\left\{-0.33197 + \frac{1}{2} \times 0.01223\right\} \\
&= 721.91
\end{aligned} \tag{24}$$

Using Equation (9) above and the bond price in Equation (24) above the yield calculation for our example month 60 (fixed rate leg) is...

$$Y(0, \tfrac{60}{12}) = \frac{\ln(1,000.00) - \ln(721.91)}{\frac{60}{12}} = 0.0652 \tag{25}$$

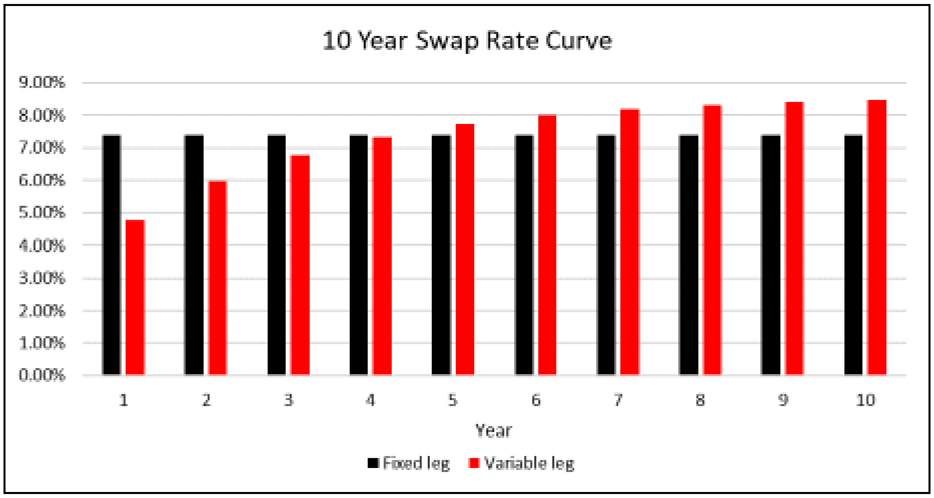
Forward rate example: Using Equations (4), (5) and (6) above and the model assumptions in the table above the price of a risk-free bond due at the end of month 48 is...

$$\begin{aligned}
\text{mean} &= 0.09 \times \frac{48}{12} + (0.09 - 0.04) \times \left(\text{Exp}\left\{-0.35 \times \frac{48}{12}\right\} - 1\right) \times 0.35^{-1} \\
&= 0.25237 \\
\text{variance} &= \frac{1}{2} \times 0.03^2 \times \left(2 \times 0.35 \times \frac{48}{12} - 3 + 4 \times \text{Exp}\left\{-0.35 \times \frac{48}{12}\right\} - \text{Exp}\left\{-2 \times 0.35 \times \frac{48}{12}\right\}\right) \times 0.35^{-3} \\
&= 0.00762 \\
\text{price} &= 1,000 \times \text{Exp}\left\{-0.25237 + \frac{1}{2} \times 0.00762\right\} \\
&= 779.92
\end{aligned} \tag{26}$$

Using Equation (18) above and the bond price equations (24) and (26) above the forward rate calculation for our example year 5 (the variable rate leg) is...

$$F(0, \tfrac{48}{12}, \tfrac{60}{12}) = \frac{\ln(779.92) - \ln(721.91)}{\frac{60}{12} - \frac{48}{12}} = 0.0773 \tag{27}$$

Using the example calculations above as our guide the graph of the swap rate curve over the time interval $[0, 10]$ is...



References

[1] Gary Schurman, *The Vasicek Interest Rate Process: Part I - The Short Rate,* February, 2013.

[2] Gary Schurman, *The Vasicek Interest Rate Process: Part II - The Stochastic Discount Rate,* March, 2013.

[3] Gary Schurman, *The Vasicek Interest Rate Process: Part III - Bond Price Equation,* March, 2013.

Appendix

A. We want to calculate the derivative of the log of bond price with respect to the time variable t. Using Equation (6) above the equation for bond price is...

$$B(s,t) = FV\,\mathrm{Exp}\left\{-\int_s^t r_x\,\delta x\right\} \tag{28}$$

Taking the log of both sides of Equation (28) above the equation for the log of bond price is...

$$\ln(B(s,t)) = \ln(FV) - \int_s^t r_x\,\delta x \tag{29}$$

Taking the derivative of Equation (29) above the equation for the derivative of the log of bond price with respect to the time variable t is...

$$\frac{\delta}{\delta t}\ln(B(s,t)) = -r_t \tag{30}$$